Math 279 Lecture 24 Notes

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1 Setup for Solving the KPZ Equation

1.1 Kernel of the KPZ equation

Last time, we showed that if $u \in \mathcal{C}^\alpha$, then $u * K \in \mathcal{C}^{\alpha+\beta}$, where K is a function that has the following properties:

(i) $\operatorname{supp} K \subseteq B_1(0)$, and K is smooth off of 0.

(ii) $|\partial^k K(x)| \lesssim |x|^{\beta-d-|k|}$.

For example, when K is the kernel of $(-\Delta)^{-1}$ and $d \geq 3$, then we have our estimate for $\beta = 2$, except that its kernel $c_0|x|^{2-d}$ is not of compact support. However, we can express our kernel as $K + \widehat{K}$, with K as above and $\widehat{K}$ a smooth function so that $u * \widehat{K}$ is smooth. Moreover, instead of convolution, we can also integrate againsta kernel $K(x,y)$, and for our Schauder estimate, we need K to behave smoothly away from the diagonal, and near the diagonal as above.

For our KPZ equation, we need a Schauder estimate for the operator $(\partial_t - \Delta)^{-1}$. Its kernel, $K(x,t) := (4\pi t)^{-d/2} e^{-|x|^2/(4t)} \mathbb{1}_{\{t>0\}}$ does not look like what we have had so far. Though we can achieve a similar claim with identical proof, provided that we follow the parabolic scaling, treating time as 2.

For one thing, we may use the metric

$$d((x,t),(y,s)) = |(x-y, t-s)|_{\mathrm{par}} = |x-y| + \sqrt{|t-s|}$$

and denote

$$\widetilde{\varphi}^\delta_{(y,s)}(x,t) = \frac{1}{\delta^{d+2}} \varphi(\tfrac{x-y}{\delta}, \tfrac{t-s}{\delta^2}),$$

where the $\sim$ means that we are using parabolic scaling. We can also discuss the size of a multiindex by

$$k = (k_1, \dots, k_d, \underbrace{k_{d+1}}_{\text{time variable}}), \qquad |k|_{\mathrm{par}} = k_1 + \cdots + k_d + 2k_{d+1}.$$

With these conventions, we may take a kernel $K(x,t)$ and assume

$$|\partial^k K(x,t)| \lesssim |(x,t)|_{\text{par}}^{\beta-(d+2)-|k|_{\text{par}}}.$$

Moreover, if $\alpha < 0$, then $\widetilde{\mathcal{C}}^\alpha(\mathbb{R}^{d+1})$ would consist of distributions F such that

$$[F]_{\alpha,K} = \sup_{(x,t)\in K} \sup_{\delta\in(0,1]} \sup_{\varphi\in D_r} \frac{|F(\widetilde{\varphi}^\delta_{(x,t)})|}{\delta^\alpha} < \infty.$$

In particular, we will have our Schauder estimate for such a kernel K, in the sense that if $u \in \widetilde{\mathcal{C}}^\alpha$, then $K * u \in \widetilde{\mathcal{C}}^{\alpha+\beta}$.

For example, the bound above holds for the heat kernel $K(x,t) = (4\pi t)^{-d/2} e^{-|x|^2/(4t)} \mathbb{1}_{\{t>0\}}$ for $\beta = 2$. Here are some details:

$$t^{-d/2} e^{-\frac{1}{4}(\frac{|x|}{\sqrt{t}})^2} \lesssim (|x| + \sqrt{t})^{2-(d+2)} = (|x| + \sqrt{t})^{-d} = \left(\frac{|x|}{\sqrt{t}} + 1\right)^{-d} t^{-d/2}.$$

This is equivalent to

$$e^{-z^2/4} \lesssim (z+1)^{-d}, \quad \text{or} \quad (z+1)^d \lesssim e^{z^2/4}.$$

Taking $\frac{d}{dt}$ gives

$$t^{-d/2-1} e^{-\frac{1}{4}(\frac{|x|}{\sqrt{t}})^2} \lesssim (|x| + \sqrt{t})^{-d-2} = (\sqrt{t})^{-d-2} \left(\frac{|x|}{\sqrt{t}} + 1\right)^{-d-2}.$$

Then we can expand the left hand side to get that

$$\frac{t^{-d/2}}{t} \frac{|x|^2}{t} e^{-\frac{1}{4}(\frac{|x|}{\sqrt{t}})^2} \lesssim (\sqrt{t})^{-d-2} \left(\frac{|x|}{\sqrt{t}} + 1\right)^{-d-2}.$$

Then perform induction.

1.2 Regularity considerations for white noise

Return to the KPZ equation

$$\begin{cases} h_t = \Delta h + |h_x|^2 + \xi \\ h(x,0) = h^0(x), \end{cases}$$

which can be written as

$$h = K * h^0 + K * (|h_x|^2 + \xi),$$

where ξ is the white noise. Let us examine the regularity of ξ. Recall that $\xi(x,t)$ is Gaussian with

$$\mathbb{E}[\xi(\varphi)] = 0, \qquad \mathbb{E}[(\xi(\varphi))^2] = \int \varphi^2 \, dx \, dt.$$

Hence,

$$\begin{aligned}
\mathbb{E}[(\xi(\widetilde{\varphi}^{\delta}_{(x,s)}))^2] &= \int (\widetilde{\varphi}^{\delta}_{(x,s)})^2 \, dx \, dt \\
&= \int \left(\frac{1}{\delta^{d+2}}\right)^2 \varphi(\tfrac{y-x}{\delta}, \tfrac{t-s}{\delta}) \, dt \, dy \\
&= \delta^{-(d+2)} \int \varphi^2.
\end{aligned}$$

We learn that

$$(\mathbb{E}[|\xi(\widetilde{\varphi}^{\delta}_{(x,s)})|^2]])^{1/2} = \delta^{-(d+2)/2} \|\varphi\|_{L^2},$$

hence

$$(\mathbb{E}[|\xi(\widetilde{\varphi}^{\delta}_{(x,s)})|^{2q}])^{1/(2q)} = c_q \delta^{-(d+2)/2} \|\varphi\|_{L^2}.$$

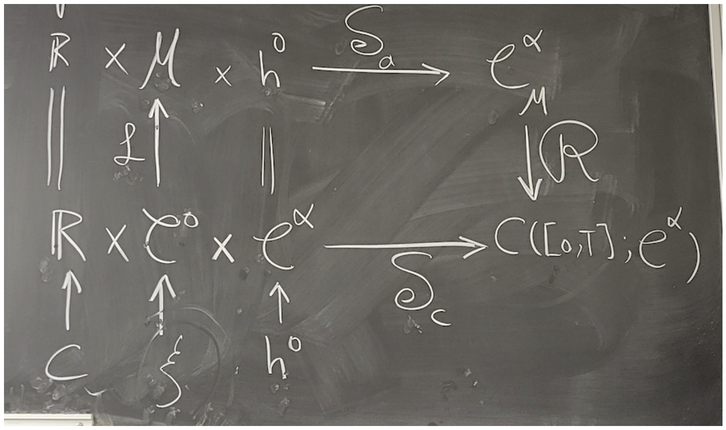
One can show that if ξ is any random Schwartz distribution with $(\mathbb{E}[(\xi(\widetilde{\varphi}^{\delta}_{(x,s)}))^{2q}])^{1/(2q)} \lesssim \delta$, then $\xi \in \widehat{\mathcal{C}}^{-\alpha-1/(2q)}$ as in Kolmogorov's theorem. Accepting this for now, we learn that $|xi \in \widetilde{\mathcal{C}}^{-(d+2)/2-\varepsilon}(\mathbb{R}^{d+1})$ for any $\varepsilon > 0$. Here, we are using parabolic scaling. As a result, we can use our Schauder estimate to assert that if K is the heat kernel, then $K * \xi \in \widetilde{\mathcal{C}}^{-d/2+1-\varepsilon} =: \widetilde{\mathcal{C}}^{-d/2+1}(\mathbb{R}^{d+1})$. For example, when $d = 1$, then $K * \xi \in \widetilde{\mathcal{C}}^{1/2-}$, which really means $\mathcal{C}^{1/2-}$ in space and $\mathcal{C}^{1/4-}$ in time.

1.3 Strategy for solving the KPZ equation

We wish to solve the KPZ equation

$$\begin{cases} h_t = \Delta h + |h_x|^2 + \xi + C \\ h(x,0) = h^0(x), \end{cases}$$

where we should really solve this as we vary the constant C. If we choose a smooth function for ξ, then we can solve this equation classically. Let us write $\mathcal{S}_c(C, \xi, h^0)$ for the classical solution. Here is the picture of what this will look like when with lift it:



Here is our strategy: We build a regularity structure that would allow us to solve the KPZ equation in abstract space, once we have a recipe for the meaning of h_x^2 so that this abstract solution is indeed a continuous operator. However, we still need to build our regularity structure. For this, let us now focus on our operator $F \mapsto F * K$, where K is the heat kernel. We claim that if our regularity structure (A, G, T) is "rich enough," then we can build an operator $\mathcal{K} : \mathcal{C}_M^\gamma \to \widetilde{\mathcal{C}}_M^{\gamma+2}$ such that

$$\mathcal{R}(\mathcal{K}f) = K * \mathcal{R}f.$$

Here, $\widetilde{\mathcal{C}}_M^\gamma = \{f : \mathbb{R}^{d+1} \to \bigoplus_{\alpha<\gamma} T_\alpha : |\Gamma_{yx} f(x) - f(y)| \lesssim |x-y|_{\text{par}}^{\gamma-\alpha}\}$, and we have the reconstruction theorem:

Theorem 1.1 (Reconstruction theorem)**.**

$$|(\mathcal{R}f - \Pi_x f(x))(\widetilde{\varphi}_x^\delta)| \lesssim \delta^\gamma.$$

As a warm-up, first let us assume that the kernel K is smooth (no singularity at 0), and assume that our regularity structure has a sector consisting of polynomials: a subspace $\overline{T}$ of T such that $T_n = \langle X^k : |k| = n \rangle$. Then, since $K * F$ is smooth for any distribution F,

$$(\mathcal{K}f)(a) = \sum_k \frac{1}{k!} (\partial^k K * \mathcal{R}f) X^k.$$

Next time, we will cover the general case.